Discrete State Space Models

In this lecture we will discuss the relation between transfer function and state space model for a discrete time system and various standard or canonical state variable models.

5.1. State Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \qquad (5.1)$$

Where;

x(k)=n-vector (state vector).

y(k)=m-vector (output vector).

u(k)=r-vector (input vector).

A=n x n matrix (input matrix).

B=n x r matrix (state matrix).

C=m x n matrix (output matrix).

D=m x r matrix (Direct transmission matrix).

Taking the Z-transform on both sides of Eqn. (5.1), we get

$$\begin{aligned} zX(z) - zx_0 &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where x0 is the initial state of the system.

$$\Rightarrow \qquad (zI - A)X(z) = zx_0 + BU(z)$$

$$\text{or,} \quad X(z) = (zI - A)^{-1} zx_0 + (zI - A)^{-1} BU(z)$$

To find out the transfer function, we assume that the initial conditions are zero, i.e., x0 = 0, thus

$$Y(z) = \left(C(zI - A)^{-1}B + D\right)U(z)$$

Therefore, the transfer function becomes

$$\boxed{G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D} \qquad (5.2)$$

which has the same form as that of a continuous time system.

5.2. Various Canonical Forms

We have seen that transform domain analysis of a digital control system yields a transfer function of the following form.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_m}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \quad m \leq n \qquad (5.3)$$

Various canonical state variable models can be derived from the above transfer function model.

5.2.1 Controllable canonical form

Consider the transfer function as given in Eqn. (5.3). Without loss of generality, let us consider the case when m = n. Let

$$\frac{X(z)}{U(z)} = \frac{1}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n}$$

In time domain, the above equation may be written as

$$\bar{x}(k+n) + \alpha_1\bar{x}(k+n-1) + \ldots + \alpha_n\bar{x}(k) = u(k)$$

Now, the output Y (z) may be written in terms of $\bar{X}$ (z) as

$$Y(z) = (\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n)\,\bar{X}(z)$$

or in time domain as

$$y(k) = \beta_0\bar{x}(k+n) + \beta_1\bar{x}(k+n-1) + \ldots + \beta_n\bar{x}(k)$$

The block diagram representation of above equations is shown in Figure 1. State variables are selected as shown in Figure 1. The state equations are then written as:

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= x_3(k) \\
\vdots &= \vdots \\
x_n(k+1) &= -\alpha_n x_1(k) - \alpha_{n-1}x_2(k) - \ldots - \alpha_1 x_n(k) + u(k)
\end{aligned}$$

Output equation can be written as by following the Figure 1.

$$y(k) = (\beta_n - \alpha_n\beta_0)x_1(k) + (\beta_{n-1} - \alpha_{n-1}\beta_0)x_2(k) + \ldots + (\beta_1 - \alpha_1\beta_0)x_n(k) + \beta_0 u(k)$$

In state space form, we have;

$$\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\
y(k) &= C\mathbf{x}(k) + Du(k)
\end{aligned} \tag{5.4}$$

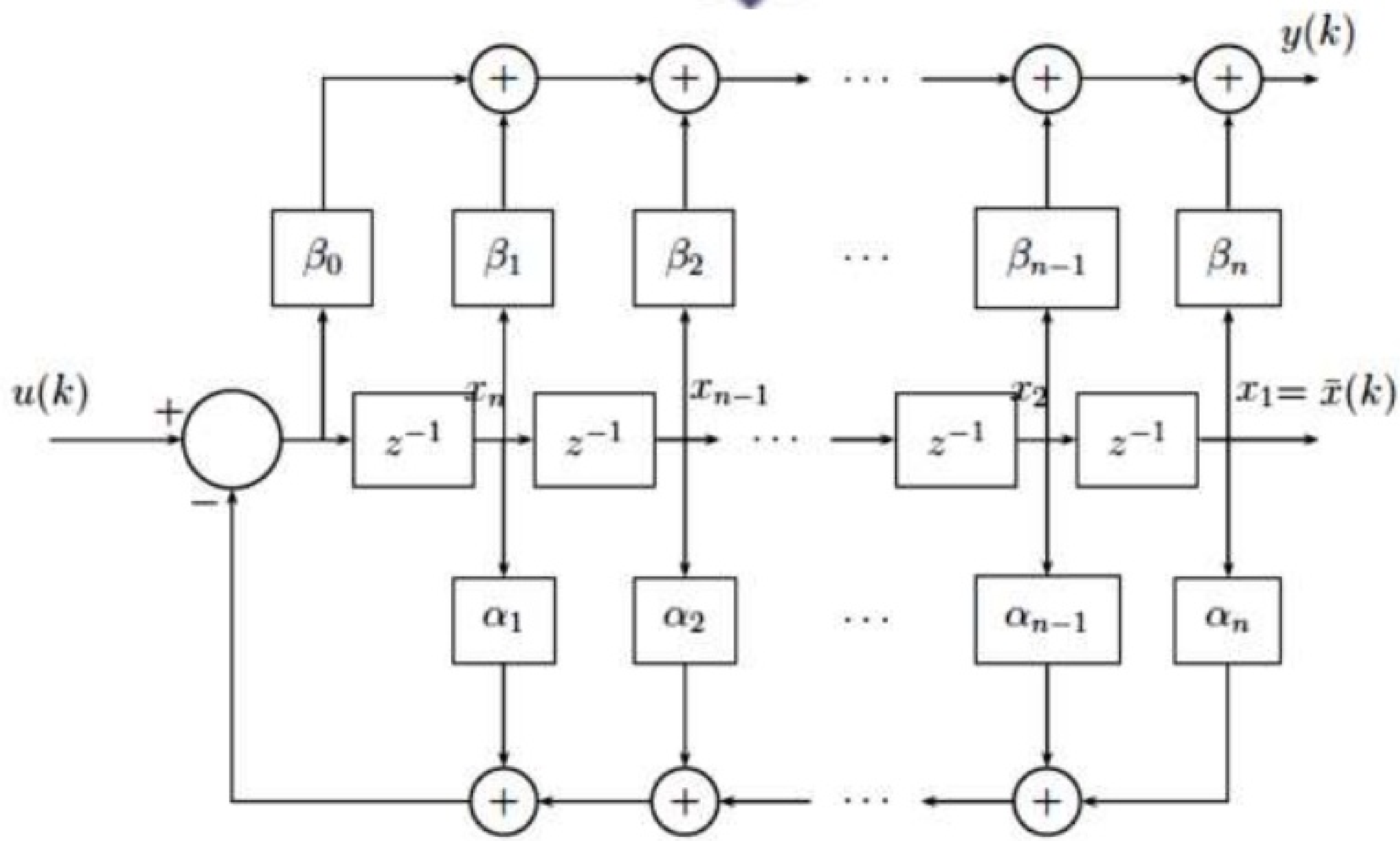

Figure 1: Block Diagram representation of controllable canonical form

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n\beta_0 \;\; \beta_{n-1} - \alpha_{n-1}\beta_0 \;\; \dots \;\; \beta_1 - \alpha_1\beta_0] \qquad D = \beta_0$$

Example 5.1:

Find the Controllable canonical form of;

$$\frac{Y(z)}{U(z)} = \frac{z + 1}{z^2 + 1.3z + 0.4}$$

Solution:

Analysing the coefficient as;

$\beta_0=0$, $\beta_1=1$, $\beta_2=1$, and $\alpha_1=1.3$, $\alpha_2=0.4$, then ;

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

5.2.2 Observable Canonical Form

Equation (3) may be rewritten as

$$\left(z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n\right) Y(z) = \left(\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n\right) U(z)$$

or, $$z^n[Y(z) - \beta_0 U(z)] + z^{n-1}[\alpha_1 Y(z) - \beta_1 U(z)] + \ldots + [\alpha_n Y(z) - \beta_n U(z)] = 0$$

or, $$Y(z) = \beta_0 U(z) - z^{-1}[\alpha_1 Y(z) - \beta_1 U(z)] - \ldots - z^{-n}[\alpha_n Y(z) - \beta_n U(z)]$$

The corresponding block diagram is shown in Figure 2. Choosing the outputs of the delay blocks

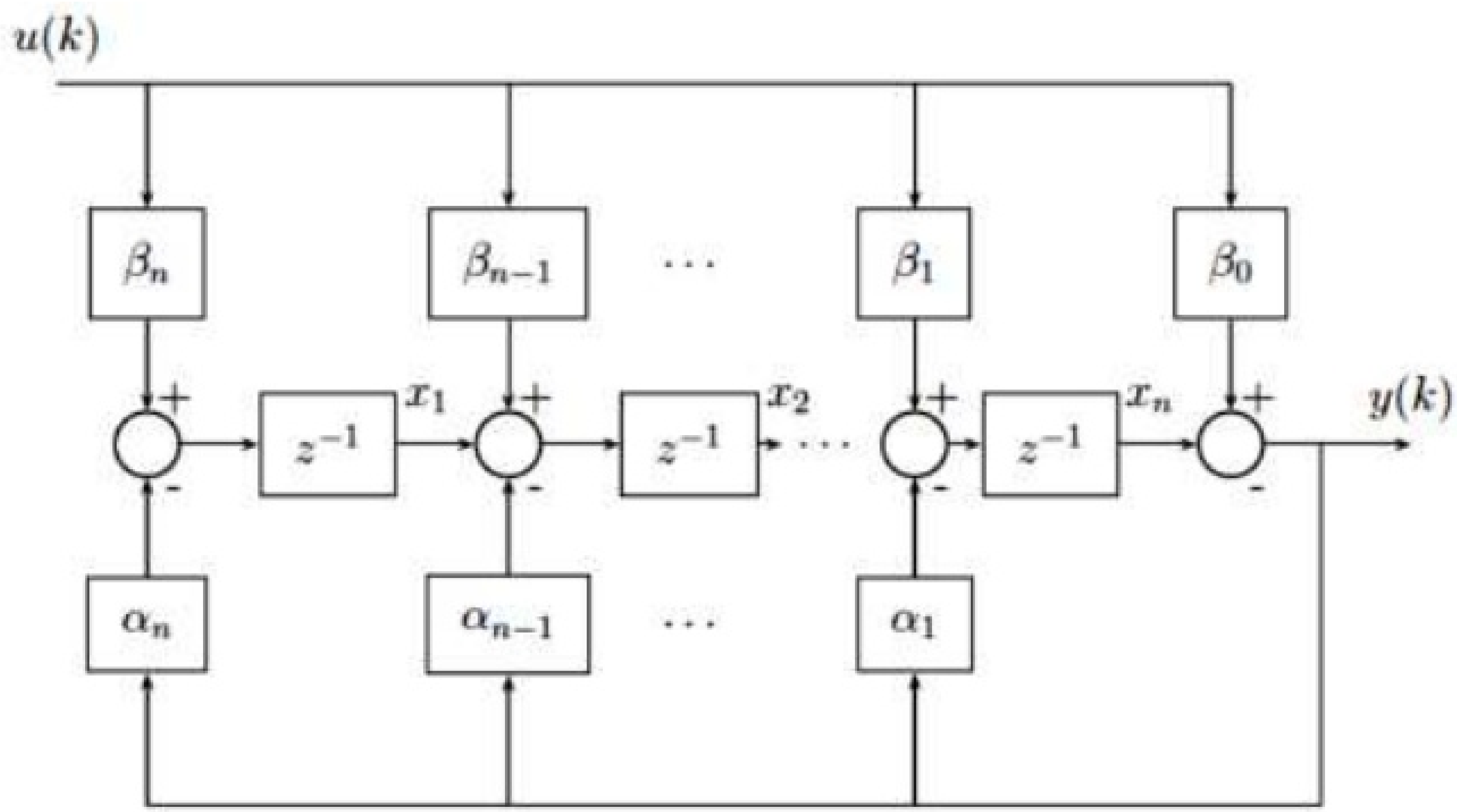

Figure 2: Block Diagram representation of observable canonical form